AP Calculus B Review Sheet
(Chapters 5-8)
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5: Logarithmic, Exponential, and Other Transcendental Functions
5.1 – The Natural Logarithmic Function and Differentiation

recalling that the Power Rule:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

does not apply when $n = -1$
such a situation is no longer an algebraic or trigonometric function, but falls into a new class called *logarithmic functions*

Definition of Natural Logarithmic Function

$$\ln x = \int_1^x \frac{1}{t}\,dt,\ \ x > 0$$

properties of the natural logarithmic function:

1. domain is 0 to infinity, range is all real numbers
2. the function is continuous, increasing, and one-to-one
3. the graph is concave down

Logarithmic Properties
if a and b are positive numbers and n is rational, then the following properties are true:
1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln(a/b) = \ln a - \ln b$

Derivative of the Natural Logarithmic Function:

let u be a differentiable function of x

1) $$\frac{d}{du}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u},\ \ u > 0$$

Derivative Involving Absolute Value:

let u be a differentiable function of x

2) $$\frac{d}{du}[\ln|u|] = \frac{u'}{u},\ \ u \neq 0$$

5.2 – The Natural Logarithmic Function and Differentiation

Log Rule for Integration

let u be a differentiable function of *x*

$$\int \frac{1}{u} du = \ln|u| + c \quad \text{also written as} \quad \int \frac{u'}{u} du = \ln|u| + c$$

- one must learn to recognize the *du/u* pattern (see ex.4 p.322)
- in some cases, the degree of *x* in the numerator may be greater than or equal to that of the denominator, but this does not exclude the possibility of a logarithmic function, since use of long division may reveal that such is actually the case (see ex.5 p.323)

Integrals of the 6 Basic Trigonometric Functions

$$\int \sin(u)\,du = -\cos u + c \qquad \int \cos(u)\,du = \sin u + c$$

$$\int \tan(u)\,du = \ln|\cos u| + c \qquad \int \cot(u)\,du = \ln|\sin u| + c$$

$$\int sec(u)\,du = \ln|sec\,u + \tan u| + c \qquad \int csc(u)\,du = -\ln|csc\,u + \cot u| + c$$

for examples of integrating trigonometric functions, see p.325-26

5.3 – Inverse Functions

Definition of Inverse Functions

a function *g* is th inverse of the function *f* if

$f(g(x)) = x$ for each *x* in the domain of *g*

and

$g(f(x)) = x$ for each *x* in the domain of *f*

the inverse function is denoted by f^{-1}

note:

if the graph of *f* contains the point (*a*,*b*), then the graph of f^{-1} contains the point (*b*,*a*)

The Existence of an Inverse Function

1. a function has an inverse if and only if it is one-to-one
2. if *f* is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse

Evaluating the Derivative of an Inverse Function

$$g(x) = \frac{1}{f'(g(x))}$$ ($g(x)$ is the inverse of $f(x)$)

one should begin by finding the value of $g(x)$

to do so, set $f(x)$ equal to the *x* input to $g(x)$

in many cases, solving for this value requires guess and check of small integers (-3 to 3)

then input this value to the first derivative of *f* and invert the result

note: graphs of inverse functions have reciprocal slopes

5.4– Exponential Functions: Differentiation and Integration

Definition of the Natural Exponential Function
The inverse of the natural logarithmic function $f(x) = \ln x$ is called the natural exponential function and is denoted by
$f^{-1}(x) = e^x$
that is $y = e^x$ if and only if $x = \ln y$

and since the function of its inverse equals x
$\ln(e^x) = x$ and $e^{\ln x} = x$

Derivative of the Natural Exponential Function
let u be a differentiable function of x

$$\frac{d}{dx}[e^u]=e^u\frac{du}{dx}$$

Integration Rules for Exponential Functions
let u be a differentiable function of x

$$\int e^u\,du = e^u+c$$

5.5 – Bases other than e and Applications

Definition of Exponential Function to Base *a*
$a^x = e^{(\ln a)x}$
note: this is derived from the definition of the inverse natural log. that states $e^{\ln x} = x$

Radioactive Half-life Model

$$amt.radioactive = (original\,amt.)(\frac{1}{2})^{\frac{t}{half\ life}}$$

Definition of Logarithmic Function to Base *a*

$$\log_a x=\frac{\ln x}{\ln a}$$

note: this was also used in algebra 2 to take the log of various bases with a calculator that only supports the common log.

Differentiation for Bases Other than *e*

$$\frac{d}{dx}[a^u]=(\ln a)a^u\frac{du}{dx}$$

$$\frac{d}{dx}[\log_a u]=\frac{1}{(\ln a)u}\frac{du}{dx}$$

Integration for Bases Other than *e*

$$\int a^x\,dx=(\frac{1}{\ln a})a^x+c$$

5.6 – Differential Equations: Growth and Decay

Separation of Variables

the equation will have y or y' on both sides of the equation and will require manipulation with integration and differentiation to solve for y (see ex.1 p.358 – *highly recommended*)

Growth and Decay Models

Exponential Growth and Decay Model

$$y = Ce^{kt}$$

C is the initial value of y, k is the proportionality constant, growth occurs when k > 1, decay occurs when k < 1

a proof exists for this formula on page 359 and may help in memorizing it

5.8 – Inverse Trigonometric Functions and Differentiation

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[arcsec\, u] = \frac{u'}{|u|\sqrt{1-u^2}} \qquad \frac{d}{dx}[arccsc\, u] = \frac{-u'}{|u|\sqrt{1-u^2}}$$

5.9 – Inverse Trigonometric Functions and Integration

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + c$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + c$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a} arcsec \frac{|u|}{a} + c$$

the remaining three inverse trigonometric functions are the negative of their corresponding function (see differentiation of inverse trigonometric functions chart)

6: Applications of Integration

6.1 – Area of a Region Between Two Curves

if *f* and *g* are continuous on [*a*, *b*] and *g(x)* ≤ *f(x)* for all x in [*a*, *b*], then the area of the region bounded above by *f(x)*, below by *g(x)*, to the left by *x* = *a*, and to the right by *x* = *b* is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

if the question asks for a region between intersecting graphs, begin by finding the points at which the graphs intersect and these will become the values of *a* and *b* in the formula.

in the case that the question asks for the area between curves that intersect at more than two points, solve for the points of intersection as before, and afterwards, calculate the area for each interval (in the case of three points of intersection, two intervals will exist, for four, three exist, etc.) and add the result. To do so, set *a* equal to the *x* value of the left-most point of intersection and *b* equal to the *x* value of the point of intersection after the previous point and use these values for the formula noting which function is greater than the other for this interval. After obtaining this value, repeat the process by setting the *x* value previously assigned to *b* as *a* and assigning the next point of intersection to *b.*

note: due to the limitations of this review, one may find the book's graphical explanations for this chapter preferable to the lengthy explanations.

6.2 – Volume: the Disk Method

used to find the volume of a three dimensional shape formed by revolving a 2 dimensional figure about the *x* axis

note: the disk method may be used to find volumes for figures revolved about the *y* axis, but this is not recommended due to the new difficulties encountered

the formula is derived by the following:

$$V = \lim_{\Delta x \to 0} \pi \sum_{i=1}^{\frac{b-a}{\Delta x}} [R(x_i)]^2 \Delta x$$

$$V = \pi \int_a^b [R(x)]^2 dx$$

R(*x*) is radius (which changes with *x*)

b - *a* represents the value of the total length of the segment around which the figure revolves

<u>Solids with Known Cross Sections</u>

The volume of any solid of which one is given a formula or value for a cross section at any given *x* value can be determined by the following:

$$V = \int_a^b A(x) dx$$

for example: if the figure contains square cross sections of varying side lengths, then $A(x) = (2R(x))^2$

6.3 – Volume: the Shell Method

used to find the volume of a three dimensional shape formed by revolving a 2 dimensional figure about the *y* axis

note: the disk method may be used to find volumes for figures revolved about the *x* axis, but this is not recommended due to the new difficulties encountered the formula is derived as follows:

$V = \text{length} \times \text{height} \times \text{width}$ // the volume of a rectangular prism

$V = 2\pi r \times h \times w$ // the volume of a single shell

$V = 2\pi \int_a^b p(x) h(x)\, dx$ // the height becomes $h(x)$ (in most cases the function of the graph), the radius becomes $p(x)$ in most cases this is simply the function which represents the x value ($p(x) = x$); the width is represented by dx

to obtain graphical explanations for this formula, it is highly recommended that one reviews the chapter in the book

6.4 – Arc Length and Surfaces of Revolution

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

<surfaces of revolution was not covered by the course>

7: Integration Techniques, L'Hopital's Rule, and Improper Integrals

7.2 – Integration by Parts

$$\int u\, dv = uv - \int v\, du$$

Guidelines for Integration by Parts (p. 481)

1. Try letting *dv* be the most complicated portion of the integrand that fits a basic integration rule. Then *u* will be the remaining factor(s) of the integrand.
2. Try letting *u* be the portion of the integrand whose derivative is a function simpler than *u*. Then *dv* will be the remaining factor(s) of the integrand.

note: *dv* may equal *dx* especially in cases where only one term is present

note: integration by parts may require repeated use

note: in the case that this method forms an integral equal to the initial, simply add/subtract it to the other side of the equation and solve for the original integral by dividing by the coefficient (see ex.5 p.485 and note the "collect like integrals")

The following is a list of suggested solution methods for for several common integrals

1. $\int x^n e^{ax}\, dx \quad \int x^n \sin ax\, dx \quad \int x^n \cos ax\, dx$
   $let\ u = x^n;\ let\ dv\ equal\ the\ remaining\ portion\ of\ the\ integral$
2. $\int x^n \ln x\, dx \quad \int x^n \arcsin ax\, dx \quad \int x^n \arctan ax\, dx$
   $let\ u = \ln x;\ let\ dv\ equal\ the\ remaining\ portion\ of\ the\ integral$
3. $\int e^{ax} \sin bx\, dx \quad \int e^{ax} \cos bx\, dx$
   $let\ u = \ln x;\ let\ dv\ equal\ the\ remaining\ portion\ of\ the\ integral$

Tabular Method

ex.7 p.486: $\int x^2 \sin 4x\, dx$

u = x²; dv = sin 4x

Alternate Signs	u & derivatives	v' & antiderivaties
(1*) +	(1) x^2	sin 4x
(2) –	(2) 2x	(1) – 1/4 cos 4x
(3) +	(3) 2	(2) –1/16 sin 4x
(4) –	(4) 0	(3) 1/64 cos 4x

*the numbers denote in which term in the subsequent polynomial the particular value belongs to (see ex.7 p.486 for a somewhat more graphical explanation)

$$\int x^2 \sin 4x\ dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32}\cos 4x + C$$

7.5 – Partial Fractions

This section will not be covered by the review at the present time. I apologize for any inconvenience this may cause. Remember to consult p.508 of your text books because this will appear on the final exam.

7.7 – Indeterminate Forms and L'Hopital's Rule

Indeterminate Forms

0/0, inf/inf, (0)(inf), 1^{inf}, 0^0, inf-inf

L'Hopital's Rule

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

Techniques for Various Indeterminate Forms

0 * inf	modify the expression to place the units that approach 0 under a fraction bar thereby converting the form to inf/inf
1^{inf}	begin by taking the natural logarithm of each side of the equation and continue by noting the new indeterminate form and solving accordingly
0^0	begin by taking the natural logarithm of each side of the equation and continue by noting the new indeterminate form and solving accordingly
inf – inf	subtract the two fractions algebraically to create an indeterminate fraction which can likely be solved with use of L'Hopital's Rule

7.4 – Improper Integrals

while previously, we were only asked to evaluate integrals from one finite number to another between which the function was continuous and finite, the ability to determine the value of indeterminate expressions permits one to overcome this limitation.

Definition of Improper Integrals with Infinite Integration Limits

the question(s) will likely ask for the value of (or the setup for) an integral of a function from either a constant to infinity (1), negative infinity to a constant (2), or negative infinity to positive infinity (3).

1. $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$
2. $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$
3. $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^{-\infty} f(x)\,dx$

Definition of Improper Integrals with Indefinite Discontinuities

the question(s) will ask for the value of (or setup for) an integral of a function which contains a vertical asymptote at one of the endpoints (1&2) or at a point between (3).

1. $\int_a^b f(x)\,dx = \lim_{c\to b^-} f(x)\,dx$
2. $\int_a^b f(x)\,dx = \lim_{c\to a^+} f(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

8: Infinite Series

Test	Series	Converges	Diverges	Comment
n^{th}–Term	$\sum_{n=1}^\infty a_n$		$\lim_{n\to\infty} a_n \neq 0$	
Geometric Series	$\sum_{n=1}^\infty ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert > 1$	$S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^\infty (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		$S = b_1 - L$
p–Series	$\sum_{n=1}^\infty \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^\infty (-1)^{n-1} a_n$	$\lim_{n\to\infty} a_n = 0$ and $0 < a_{n+1} < a_n$		

Integral	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ *converges*	$\int_1^{\infty} f(x)\,dx$ *diverges*	f is continuous, positive, and decreasing
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{x\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{x\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{x\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{x\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	
Direct Comparison	$\sum_{n=1}^{\infty} a_n$	$0 < a_n < b_n$ and $\sum_{n=1}^{\infty} b_n$ *converges*	$0 < b_n < a_n$ and $\sum_{n=1}^{\infty} b_n$ *diverges*	$a_n, b_n > 0$
Limit Comparison	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ *converges*	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ *diverges*	$a_n, b_n > 0$

The review of the remainder of chapter 8 will not be covered at the present time in the main version of the review. Please consult the condensed version and your text book to review its material.